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Stability of compacton solutions

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The stability of the recently discovered compacton solutions is studied by means of both linear stability analysis as well as Lyapunov stability criteria. From the results obtained it follows that, unlike solitons, all the allowed compacton solutions are stable, since the stability condition is satisfied for arbitrary values of the nonlinearity parameter. The results are shown to be true even for the higher order nonlinear dispersion equations for compactons. Some conservation laws for the higher order nonlinear dispersion equations are also presented. [S1063-651X(98)50909-9]

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The observed stationary and dynamical patterns in nature are usually finite in extent. However, most of the weakly nonlinear and linear dispersion equations studied so far admit solitary waves that are infinite in extent, although localized in nature. Therefore, the recently discovered compacton solutions (i.e., solitary waves with compact support) of the nonlinear dispersive $K(m,n)$ equations have become very important from the point of study of the effect of nonlinear dispersion on pattern formation as well as the formation of nonlinear structures such as liquid drops, etc. The compacton speed depends on its height, but unlike the solitons, its width is independent of the speed, a fact that seems to play a very crucial role in its stability property. Compactons have the remarkable solitonlike property that they collide elastically. However, unlike soliton collisions in an integrable systems, the point at which two compactons collide is marked by the creation of low amplitude compacton-anticompacton pairs [1,2]. In fact, it is now known that the $K(m,n)$ system of equations is not integrable [1,2]. This suggests that the observed almost-elastic collisions of the compactons are probably not due to the integrability and thus the mechanism responsible for the coherence and robustness of the compactons remains a mystery. Stability analysis of the compacton solutions may provide some clues in this direction. As has been said above, the stability of the compactons is crucial in the context of its applications in the study of pattern formation. Besides, the stability problem of the $K(m,n)$ equations is interesting because, for such equations with higher power of nonlinearity and nonlinear dispersion, the phenomenon of collapse is possible. Also, in the context of soliton equations, from the stability analysis it has been shown that the higher order linear dispersion term stabilizes the solitons [3,4]. In this regard it will be of interest to see what role the higher order nonlinear dispersion term plays with respect to the stability of the compacton solutions of the $K(m,n)$ type equations.

In this Rapid Communication we report on the stability analysis of the compacton solutions of the nonlinear dispersion $K(m,n)$ type equations as considered by Cooper *et al.* [2]. We use both the linear stability analysis and the Lyapunov stability criteria to analyze the problem. We start with the $K(l,p)$ equation:

$$u_t+u_x u^{l-2}+\alpha[2u_{3x}u^p+4pu^{p-1}u_x u_{2x}+p(p-1)u^{p-2}u_x^3]=0. \tag{1}$$

These equations have the same terms as the $K(m,n)$ equations considered by Rosenau *et al.* [1], but the relative weights of the terms are different, leading to the fact that, whereas the $K(l,p)$ equation [Eq. (1)] can be derived from a Lagrangian, the $K(m,n)$ equation considered in [1] does not have a Lagrangian. For the sake of comparison, it may be noted that the set of parameters (m,n) in [1] corresponds to the set $(l-1,p+1)$ in Eq. (1) [2]. Assuming a solution to Eq. (1) in the form of a traveling wave $u(x,t)=u(\xi)$, where $\xi=x-Dt$, Eq. (1) reduces to the same $K(l,p)$ equations considered by Cooper *et al.* [Eq. (7) in [2]] for the compacton solutions. The compacton solutions to Eq. (1) are given by [2],

$$u_c(\xi)=\left[\frac{D}{2}(p+1)(p+2)\right]^{(1/p)}\times\cos^{(2/p)}\left[\frac{p\xi}{2\sqrt{\alpha(p+1)(p+2)}}\right], \tag{2}$$

in case $l=p+2,\ 0<p\leqslant 2$, and for $|\xi|\leqslant(\pi/p)\sqrt{\alpha(p+1)(p+2)}$; $u_c(\xi)$ is zero otherwise. Note that the width of the compacton is independent of its speed (amplitude). The fact that the width of the compacton will always be independent of its amplitude in case $l=p+2$ follows from the invariance of Eq. (1) under the scaling transformation $x\to\alpha x,\ t\to\beta t,\ u\to\gamma u$. We now consider the stability of these compacton solutions.

(i) Linear stability. Equation (1) can be obtained from the variational principle $\delta(H+DP)=0$. Using the relations

$$I_n=\int_{-\infty}^{+\infty}u^n dx, \quad J_2=\int_{-\infty}^{+\infty}u^p u_x^2 dx, \tag{3}$$

we can write the corresponding conserved Hamiltonian and momentum as

$$H_c = \alpha J_2 - \frac{I_{p+2}}{(p+1)(p+2)}, \quad P_c = \frac{1}{2} I_2. \tag{4}$$

Using Eq. (1) and the equation obtained from the scaling transformation $x \to \beta x$, we get

$$\frac{I_{p+2}}{(p+1)(p+2)} = \frac{(4+p)DP_c}{2(p+2)}, \quad \alpha J_2 = \frac{pDP_c}{2(p+2)} \tag{5}$$

so that $H_c = -2DP_c/(p+2)$. Considering the general scaling transformation $u \to \mu^{1/2} u(\lambda x)$, H_c and P_c are transformed to $H(\lambda,\mu)$ and $P(\lambda,\mu)$ and we get

$$\Phi(\lambda,\mu) = \alpha\lambda\mu^{(p+2)/2} J_2 - \frac{\mu^{(p+2)/2}}{\lambda(p+1)(p+2)} I_{p+2} + \frac{\mu}{\lambda} DP_c, \tag{6}$$

where $\Phi(\lambda,\mu) = H(\lambda,\mu) + DP(\lambda,\mu)$. The equations $\partial\Phi/\partial\lambda = \partial\Phi/\partial\mu = 0$ gives the stationary point at $\lambda = \mu = 1$ (compacton equation) and near this point, using the Taylor series for $\mu = \lambda$ we get (the transformation in that case does not change the momentum P)

$$\begin{aligned}\delta^{(2)}\Phi(\lambda) &= \delta^{(2)} H(\lambda) \\ &= (\lambda-1)^2 \left[\frac{\alpha(p+2)(p+4)J_2}{8} \right. \\ &\quad \left. - \frac{p(p-2)(p+4)DP_c}{16(p+2)} \right],\end{aligned} \tag{7}$$

which has a definite sign. If it is positive (negative) the expression

$$H(\lambda) = \alpha\lambda^{[(p+4)/2]} J_2 - \frac{\lambda^{(p/2)}(p+4)DP_c}{2(p+2)} \tag{8}$$

has a minimum (maximum) at $\lambda = 1$.

Now, let us assume that $u = u_c + v$, where $|v| \ll 1$ and the scalar product $(u_c, v) = 0$. Substituting this in Eq. (1) after linearization we get

$$\partial_T v = \partial_\xi \hat{L} v, \tag{9}$$

where $\xi = x - Dt$ and $T = t$ and the operator $\hat{L}$ is given by

$$\begin{aligned}\hat{L} = [&D - u^{l-2} - 2\alpha p u^{p-1} u_{2\xi} - 2\alpha u^p \partial_\xi^2 \\ &- \alpha p(p-1) u^{(p-2)} u_\xi^2 - 2\alpha p u^{(p-1)} u_\xi \partial_\xi].\end{aligned} \tag{10}$$

One can now run through the steps as given by Karpman [3] [see Eqs. (29)–(32)] and show that as in his case, in our case the sufficient condition for stability is that the scalar product $(\psi, \hat{L}\psi) > 0$, where the operator $\hat{L}$ is given by Eq. (10), while ψ is a function in the subspace orthogonal to u_c. However, condition $(\psi, \hat{L}\psi) > 0$ is also associated with the extremum of $H + DP$ since, using the relation $\delta(H+DP) = 0$, one can show that the second variation of $H(u)$ and $P(u)$ at $u = u_c$ is given by

$$\delta^{(2)}(H+DP)_{u_c} = \frac{1}{2} \int_{-\infty}^{+\infty} (v, \hat{L} v) d\xi > 0, \tag{11}$$

where the operator $\hat{L}$ is given by Eq. (10). That is, if the condition $(\psi, \hat{L}\psi) > 0$ is fulfilled, then $H(u) + DP(u)$ has a minimum at $u = u_c$. Inversely, the minimum of $H(u) + DP(u)$ at $u = u_c$ is a sufficient condition of compacton stability with respect to small perturbation. Using Eq. (8) we obtain the condition for the minimum of the perturbed Hamiltonian $H(\lambda)$ at $\lambda = 1$ as $p+2 > p-2$, which is obviously true for any p. Thus, we see that the condition for the Hamiltonian minimum (and hence the sufficient condition for the compacton stability) is satisfied for arbitrary values of the nonlinear parameter p. This is unlike the soliton stability results, where it has been shown that the stability condition puts a restriction on the allowed values of the nonlinear parameter [3,4,6]. Note, however, that compacton solutions exist only for $p \leqslant 2$.

In the same-way as Karpman [3], one can show that the sufficient condition $(\psi, \hat{L}\psi) > 0$ is also equivalent to the condition

$$\left(\frac{\partial P_c}{\partial D} \right) > 0. \tag{12}$$

From Eqs. (2) and (4) one can easily show that the sufficient condition for compacton stability [Eq. (12)] is satisfied for arbitrary values of the nonlinearity parameter p. It should be noted that this result is completely in contrast to the usual soliton stability results. It is not difficult to see why this is so for the compactons. From Eq. (2) we see that the width of the compacton solutions is independent of its speed (amplitude) D, and the generic form of such compactons is $u_c(\xi) = AD^b \cos(c\xi)$, where the constants A, b, and c depend on the nonlinearity parameter. Hence $P_c = D^{2b} K$, where K is D independent. Therefore, $dP/dD > 0$ trivially since $b > 0$. On the other hand, if the width depends on speed (as in the case of solitons [3,4]) with the generic form of the solution as $u(\xi) = AD^b \cos(cD^a \xi)$, then $dP/dD > 0$ only if $2b > a$, which will depend on the particular theory (soliton equations). It should be noted that the above stability condition [Eq. (12)] is obtained by assuming that there is only one -ve eigenvalue for the operator $\hat{L}$ [Eq. (10)]. The validity of this conjecture has been proven from numerical experiments for many other systems, such as the third and fifth order Korteweg–de Vries equations as well as the nonlinear Schrödinger equations [3]. At present we do not have any evidence to show that this conjecture is also valid for our operator $\hat{L}$ [Eq. (10)], except for the fact that the result that follows from using this conjecture also agrees with the result obtained from the Hamiltonian minimum condition [Eq. (11)], as well as an independent analysis of the stability by the Lyapunov method as shown below.

These results are also true for the higher order dispersion equations. For example, consider the fifth order nonlinear dispersion equations $K(m,n,p)$ [5,7] for the compactons. The Hamiltonian for such system is, respectively, given by [5]

$$H_c = \int_{-\infty}^{+\infty} \left[\delta \frac{u^{m+1}}{(m+1)} + \alpha u^{n-1} u_x^2 + \beta u^{p-3} u_x^4 + \gamma u^{p-1} u_{2x}^2 \right] dx \tag{13}$$

and [7]

$$H_c=\int_{-\infty}^{+\infty}\left[\beta u^m u_x^2-\alpha\frac{u^{p+2}}{(p+1)(p+2)}-\frac{\gamma}{2}u^n u_x^l u_{2x}^2\right]dx. \tag{14}$$

As has been shown in [5], the compacton solutions corresponding to the Hamiltonian in Eq. (13) are allowed for the nonlinearity parameter $k=m=n=p$ in the range $2\leqslant k\leqslant 5$, thereby meaning that the effect of the higher order nonlinear dispersion term is to increase the range of the nonlinearity parameter for which the compacton solutions are allowed. Considering the small perturbation $u=u_c+v$ as before, we can show that [8] even for these higher order nonlinear dispersion equations, the sufficient conditions for the compacton stability as given by Eqs. (11) and (12) are satisfied for arbitrary values of the nonlinear parameter k.

(ii) Lyapunov stability. The above theory of linear stability analysis for the compactons is based on the linearization of equations for compacton perturbations. This method has some inherent limitations connected with the linearization. Therefore, we present another approach to the stability problem based on the Lyapunov method, which, instead of linearization, uses sharp estimates. The effectiveness of this method has been demonstrated by Weinstein [9] and Karpman *et al.* [10]. In this method of analysis, it is sufficient to prove that the Hamiltonian is bounded from below for fixed momentum P and the compacton realizes the Hamiltonian minimum. Here we consider the stability of the compacton solution of Eq. (1). From Eq. (3), we have

$$I_{p+2}\leqslant(\max u^{(p+4)/2})^{(2p)/(p+4)}\int u^2dx. \tag{15}$$

Also,

$$\max(u^{(p+4)/2})\leqslant\frac{p+4}{2}\int |u^{p/2}u_x||u|dx. \tag{16}$$

Using Holder's inequality, we get

$$\max(u^{(p+4)/2})\leqslant\frac{p+4}{2}\left[\int u^p u_x^2dx\right]^{1/2}\left[\int u^2dx\right]^{1/2}$$
$$\leqslant\frac{p+4}{2}J_2^{1/2}(2P)^{1/2}. \tag{17}$$

From Eqs. (4) we then have

$$H\geqslant\min_{J_2}\left[\alpha J_2-\frac{1}{(p+1)(p+2)}\left(\frac{p+4}{2}\right)^{(2p)/(p+4)}\times J_2^{p/(p+4)}(2P)^{(2p+4)/(p+4)}\right]. \tag{18}$$

Thus H is bounded from below. On calculating the minimum of the right hand side, we find $H_{\min}=-(4/p)\alpha J_2$ Thus we see that H is bounded from below for arbitrary values of the nonlinearity parameter p. Now, from Eqs. (4) and (5) we can immediately see that $H_c=H_{\min}=-(2DP_c)/(p+2)$, i.e., the compacton realizes the Hamiltonian minimum and hence this proves the stability of the compacton solutions in the Lyapunov sense. Thus, the Lyapunov stability analysis shows that all the allowed compacton solutions (i.e., $p\leqslant 2$) are stable, since the condition for boundedness of the Hamiltonian and $H_c=H_{\min}$ are valid for arbitrary values of the nonlinearity parameter p. It can be shown that [8] we also get similar results from the Lyapunov stability analysis of the higher order nonlinear dispersion equation as given by the Hamiltonians in Eqs. (13) and (14).

Before concluding, we would like to mention that in our earlier paper [5] we had reported only one conservation law for the higher order $K(m,n,p)$ equations given by [Eq. (2) in [5]]

$$u_t+\beta_1(u^m)_x+\beta_2(u^n)_{3x}+\beta_3(u^p)_{5x}=0,\quad m,n,p>1. \tag{19}$$

We now find that, like the $K(m,n)$ equations as considered by Rosenau *et al.* [1], the higher order $K(m,n,p)$ equations also have four conservation laws for $m=n=p$, with the same conserved quantities as for the $K(m,n)$ equations, i.e.,

$$Q_1=u,\quad Q_2=u^{m+1},\quad Q_3=u\cos x,\quad Q_4=u\sin x. \tag{20}$$

We have checked that even for the seventh order nonlinear dispersion $K(m,m,m,m)$ equation [see Eq. (53) in [5]] there are four conservation laws as above. We suspect that even the generalized arbitrary odd-order nonlinear dispersion $K(m,m,m,m,\ldots)$ equations [see Eq. (36) in [5]] may also support similar four conservation laws. However, it should be noted that for the $K(l,p)$ equation [Eq. (1)] and its corresponding higher order $K(m,n,p)$ equation [Eqs. (13), (14)], which are derivable from a Lagrangian and whose stability problem is considered here, there are only three conservation laws [2,5].

To conclude, we would like to point out the important difference between the soliton and compacton solutions as obtained from the stability analysis of such solutions. Whereas the soliton solutions are allowed for arbitrary values of the nonlinear parameter, the stability condition on the soliton solutions puts restrictions on the nonlinearity parameter for which stable soliton solutions are allowed [3,4,6,10]. On the other hand, the compacton solutions are allowed only within a certain range of the nonlinear parameter (the range is determined from the condition of the finite derivative of the compacton solutions at the edges [2,5]) and all the allowed compacton solutions (within this allowed range of the nonlinear parameter) are stable. Unlike soliton solutions, the stability of the compacton solutions does not put any additional constraint on the range of the nonlinear parameter. This result is true even for the higher order nonlinear dispersion equations for compactons, whereas for the soliton case the higher order linear dispersion term stabilizes the solitons with a higher power of nonlinearity. It may be noted that we are unable to discuss the question of the stability of the compacton solutions as considered by Rosenau *et al.* [1] since their compacton equations cannot be derived from a Lagrangian. However, we suspect that their compacton solutions will also be stable. It would be nice if this could be shown in general.

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